J. Appl. Maths Mechs, Vol. 55, No. 1, pp. 106-113, 1991
Printed in Great Britain
0021-8928/91 $15.00+0.00

INTEGRAL CHARACTERISTICS OF SOLUTIONS OF SPATIAL PROBLEMS ON THE DYNAMICAL IMPRESSION OF SOLID BODIES IN CONTINUOUS MEDIA*

F.M. BORODICH

Spatial problems of the impression of arbitrary bodies into a half-space occupied by a continuous medium are examined in a geometrically linear formulation. It is shown that if the governing relationships are linear, while the medium is homogeneous inhomogeneous only in depth, then in the initial interaction stage the problem of determining the integral characteristics of the solutions (the integral displacements and resultant forces) is equivalent to the problem of plane-wave propagation in this same medium. Expressions are obtained for the interaction forces between the body and the medium (resultant forces) in a number of specific cases: a non-linear elastic medium with initial stresses, viscoelastic media, and an isotropic elastic medium, smoothly inhomogeneous in depth. It is shown that all the results hold for both vertical impression and for impression with rotation.

Expressions for the resultant forces have been obtained earlier by other methods in the following problems: vertical impression for an acoustic medium /1/ and an isotropic elastic medium /2, 3/, impression with rotation for an isotropic elastic medium /4/ and vertical impression in an anisotropic elastic medium /5/.

1. Formulation of the problem. We consider a continuous medium in which the connection between the stress tensors σ and the small strains ε is given in the form

$$\sigma = F(\varepsilon),\ \varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \tag{1.1}$$

where F is some operator, and u_i is the displacement vector component. We will assume that the velocity of perturbation propagation in this medium is finite while the medium itself occupies the positive half-space $\mathbf{R}_+^3$.

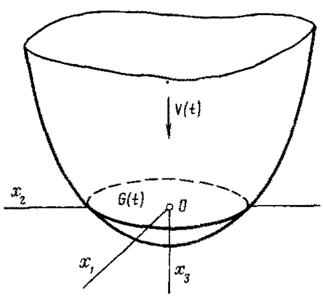

Fig.1

Let a smooth blunt body, whose shape is determined by the graph of a non-negative function $f(x_1, x_2)$ be impressed in a half-space $\mathbf{R}_+^3$ initially at rest. We select the origin of a Cartesian system of coordinates at the point of initial body contact with the medium. We direct the x_3 axis into the depth of the half-space, and the x_1 and x_2 axes along its boundary. We first examine the vertical impression with a positive velocity $V(t)$ (Fig.1). Then the depth of impression of the body apex H is found from the formula

$$H(t) = \int_0^t V(\lambda)\, d\lambda \tag{1.2}$$

We take a section of the body surface at a height $H(t)$ and project it on to the $x_3 = 0$ plane. The velocity of propagation of the boundary of this projection at the point x_1, x_2 equals $V(t)\,|\operatorname{grad} f(x_1, x_2)|^{-1}$. On the blunt body $|\operatorname{grad} f(0, 0)| = 0$. Hence, a time interval exists for any finite velocity of perturbation propagation in the medium at which the propagation velocity of the boundary of the projection of the body section exceeds the maximum velocity of perturbation propagation in the medium a.

Let G be a domain of body contact with the medium, ∂G its boundary, and $\gamma(x_1^*, x_2^*, t)$, $(x_1^*, x_2^*) \in \partial G$ the velocity of motion ∂G. We will consider the problem of the time interval $[0, T]$ on which the following equality holds

$$\gamma(x_1, x_2, t) = V(t)\,|\operatorname{grad} f(x_1, x_2)|^{-1} \tag{1.3}$$

If the initial velocity $V(0)$ is not zero, then the quantity T is non-zero. It is said that the impression process in the interval $[0, T]$ is superseismic in nature. The

*Prikl.Matem.Mekhan.,55,1,133-141,1991

equality (1.3) is known to be satisfied if the velocity $\gamma(x_1, x_2, t)$ is greater than the quantity a (and can still be satisfied a certain time because the velocity of perturbation propagation in the direction in the horizontal plane and orthogonal to the body surface can be less than the maximum value).

Example. Let a paraboloid of revolution be impressed into a medium with constant velocity V_0, i.e.,

$$f(x_1, x_2) = Ar^2, \quad r = \sqrt{x_1^2 + x_2^2}, \quad A > 0, \quad V(t) \equiv V_0$$

Then from the condition

$$T \geqslant t_1, \gamma(x_1^*, x_2^*, t_1) = a$$

we obtain the following estimate $T \geqslant V_0/(4a^2A)$.

It is obvious that for $t \in [0, T]$ at the points $\partial G(t)$ the component u_3 of the displacement vector will experience a break, i.e., u_3 is not smooth. Consequently, we will consider a weak solution of the problem.

We consider $\mathbf{u}$ and σ to satisfy the conditions

$$\int_0^T \int_{\mathbf{R}_+^3} (\rho\varphi_\alpha \partial^\cdot \partial^\cdot u_\alpha - \varphi_{\alpha,j}\sigma_{\alpha j} + \varphi_\alpha X_\alpha)\, d\mathbf{v}\, dt = 0 \tag{1.4}$$

$$\forall \varphi \in C_0^1(\mathbf{R}_+^3 \times [0, T]), \quad \mathbf{u} \in W_2^1(\mathbf{R}_+^3 \times [0, T])$$

Here and henceforth, the comma before the subscript will denote the derivative with respect to the corresponding coordinate; ∂_i and $\partial^\cdot$ are generalized derivatives (in the Sobolev sense) with respect to the coordinates and time, respectively; summation from 1 to 3 is assumed over repeated Latin subscripts, while there is no summation over the Greek subscripts; W_2^1 is the Sobolev space, C_0^1 is a set of continuously differentiable functions that vanish on the boundary of the domain of definition and have compact support, ρ is the density of the medium, and X_α is the component of the volume force field.

We will formulate an initial-boundary value problem on dynamic impression. We will seek $\mathbf{u}$ and σ connected by the conditions (1.1) (in which the derivatives are replaced by the generalized derivatives) and satisfying conditions (1.4). We consider the function $\mathbf{u}$ to be continuous and to satisfy the generalized initial and boundary conditions

$$\int_0^T \int_{\mathbf{R}_+^3} (\rho\varphi_\alpha \partial^\cdot \partial^\cdot u_\alpha - \varphi_{\alpha,j}\sigma_{\alpha j} + \varphi_\alpha X_\alpha)\, d\mathbf{v}\, dt + \int_{\mathbf{R}_+^3} \rho(\mathbf{x})\varphi_\alpha(\mathbf{x}, 0) v_\alpha^0(\mathbf{x})\, d\mathbf{v} +$$
$$\int_0^T \int_{\mathbf{R}^2} \varphi_\alpha T_\alpha\, dx_1\, dx_2\, dt = 0, \quad \alpha = 1, 2, 3$$
$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \; u_3(x_1, x_2, 0, t) = g(x_1, x_2, t), \quad (x_1, x_2) \in G(t)$$

Here $\mathbf{u}^0$ and $\mathbf{v}^0$ are the initial displacements and velocities of points of the medium, respectively, T_1 and T_2 are tangential forces given on the whole boundary plane, T_3 and normal forces given outside the contact domain, g is a function known in the domain $G(t)$, and the integral identity is satisfied for any functions φ such that

$$\varphi \in C^1(\mathbf{R}_+^3 \times [0, T]), \; \varphi(\mathbf{x}, T) = 0, \; \varphi_3(x_1, x_2, 0, t) = 0, \; (x_1, x_2) \in G(t), \; 0 \leqslant t \leqslant T$$

In the problem under consideration $\mathbf{u}^0 \equiv \mathbf{v}^0 \equiv T_1 \equiv T_2 \equiv 0$, $T_3(x_1, x_2, t) = 0$, $(x_1, x_2) \in \mathbf{R}^2 \setminus G(t)$ and the function $g(x_1, x_2, t)$ is known on the whole boundary plane of the half-space in the superseismic stage. Consequently, the initial and boundary conditions can be written in the following form

$$\int_0^T \int_{\mathbf{R}_+^3} (\rho\varphi_\alpha \partial^\cdot \partial^\cdot u_\alpha - \varphi_{\alpha,j}\sigma_{\alpha j} + \varphi_\alpha X_\alpha)\, d\mathbf{v} dt = 0 \tag{1.5}$$

$$\mathbf{u}(\mathbf{x}, 0) = 0, \; u_3(x_1, x_2, 0, t) = g(x_1, x_2, t)$$

The integral identity is satisfied for all functions φ such that

$$\varphi \in C^1(\mathbf{R}_+^3 \times [0, T]), \quad \varphi(\mathbf{x}, T) = 0, \quad \varphi_3 \in C_0^1(R_+^3), \quad 0 \leqslant t \leqslant T$$

We will consider that the conditions of the uniqueness theorem are satisfied for the medium under consideration (uniqueness theorems for weak solutions for acoustic, elastic, and viscoelastic media are considered in a number of publications /6-8/).

2. ***The integral characteristics of solutions of dynamic problems.*** We introduce the following integral characteristics

$$\mathbf{w}(x_3, t) = \iint \mathbf{u}\,dx_1dx_2, \quad \mathbf{g}_i(x_3, t) = \iint \partial_i\mathbf{u}\,dx_1dx_2 \tag{2.1}$$
$$\mathbf{g}_4(x_3, t) = \iint \partial^{\cdot}\mathbf{u}\,dx_1dx_2, \quad \Sigma_{ij}(x_3, t) = \iint \sigma_{ij}\,dx_1dx_2$$

where integration in the x_1x_2 plane is performed everywhere in the domain $\mathbf{R}^2$.

Let $\mathbf{u}$ and σ satisfy conditions (1.4) and let the support of these functions be limited for $0 \leqslant t \leqslant T$. Then the following assertion holds (see /9/).

Lemma. The integral characteristics introduced in (2.1) possess the following properties:

1) $\mathbf{w}$, $\mathbf{g}_i$, $\mathbf{g}_4$ are determined almost everywhere in $R_+ \times [0, T]$ and are summable;

2) $\mathbf{g}_3$ and $\mathbf{g}_4$ are generalized derivatives of the function $\mathbf{w}(x_3, t)$ with respect to x_3 and t, respectively;

3) $\mathbf{g}_1$ and $\mathbf{g}_2$ equal zero for almost all x_3 and t;

4) $\mathbf{w}(x_3, t) \in W_2^1(R_+ \times [0, T])$;

5) $\mathbf{g}_3$ and $\mathbf{g}_4$ are functions from $L_2(\mathbf{R}_+ \times [0, T])$.

Let $\mathbf{u}$ and σ be a weak solution of problem (1.1), (1.4) and (1.5). Then by virtue of the finiteness of the velocity of perturbation propagation in the medium and the boundedness of the supports of the functions $\mathbf{u}$ and σ at the initial time the supports of the functions $\mathbf{u}$ and σ are bounded for any finite t. Thus, the conditions of the lemma are satisfied.

Let $\mathbf{X}$ and ρ be independent of x_1 and x_2.

Assertion 1. The vector $\mathbf{w}$ and the tensor Σ_{ij} introduced in (2.1) satisfy the following initial-boundary value problem

$$\mathbf{w} \in W_2^1(R_+ \times [0, T]); \quad \mathbf{w} \in C(R_+ \times [0, T]) \tag{2.2}$$

$$\mathbf{w}(x_3, 0) = 0; \; w_3(0, t) = Y(t), \; Y(t) = \iint g(x_1, x_2, t)dx_1, dx_2$$
$$\int_0^T \int_{R_+} (\rho\psi_\alpha^{\cdot}\partial^{\cdot}w_\alpha - \psi_{\alpha,3}\Sigma_{\alpha 3} + \psi_\alpha X_\alpha)\,dx_3dt = 0$$

The last identity is satisfied for any functions ψ such that

$$\psi(x_3, t) \in C^1(R_+ \times [0, T]), \quad \psi(x_3, T) = 0, \quad \psi_3 \in C_0^1(R_+), \quad 0 \leqslant t \leqslant T$$

Proof. We take $\mathbf{u}$ and σ, the solution of problem (1.1), (1.4) and (1.5) and construct integral characteristics for it. Applying the lemma we see the validity of the assertion.

We note that in the general case if the operator F in (1.1) is non-linear, then it is impossible to set up a connection between the components of the tensor Σ_{ij} and the vector $\mathbf{w}$; if the operator F is linear, then the components of Σ_{ij} are defined uniquely with respect to $\mathbf{w}$.

Theorem. If the continuous medium is homogeneous in the coordinates x_1, x_2 and the relationships (1.1) are linear, and also if the functions $\mathbf{X}$ and ρ are independent of the coordinates x_1 and x_2, then the problem of finding the integral characteristics of the solution of the dynamical problem (1.1), (1.4) and (1.5) on the impression of a blunt body in a half-space $\mathbf{R}_+^3$ filled by this medium is equivalent to the initial-boundary value problem (2.2) concerning plane-wave propagation in this medium.

Proof. We take the solution of the problem under consideration and by using (2.1) we introduce the vector $\mathbf{u}$ and tensor σ and by direct substitution we see that the theorem is valid.

Remark. Let $\mathbf{v}$ be a vector of the external normal to the surface $x_3 = 0$. Then an integral force vector $\mathbf{N}$: $N_i = \Sigma_{ij}v_j$ can be introduced, and the interaction force between the body and medium will equal

$$P = -\langle \mathbf{N}(0, t), \mathbf{v}\rangle \equiv -\Sigma_{33}(0, t) \tag{2.3}$$

We consider below the case of impression in specific media for $X_\alpha \equiv 0$.

3. ***A medium with initial stresses.*** Let a non-linearly elastic medium be prestressed by homogeneous forces at infinity. We consider the stresses caused by body contact with the medium to be small compared with the initial stress. If Cartesian coordinates x_1, x_2, x_3 of the initial deformed state are introduced /10/, then the equations of motion and the linearized governing relationships will have the form

$$Q_{ij,i} - \rho u_j^{\cdot\cdot} = 0, \quad Q_{ij} = \omega_{ijkl} u_{k,l} \tag{3.1}$$

where Q_{ij} are components of the non-symmetric "true" stress tensor while ω_{ijkl} are components of the constant (by virtue of homogeneity) fourth-rank tensor. The components ω_{ijkl} possess the following symmetry properties /10/

$$\omega_{ijkl} = \omega_{lkji} \tag{3.2}$$

We introduce the matrix $\Lambda_{ij} = \omega_{3ij3}/\rho$ symmetric because of (3.2). It follows from the Hadamard theorem /10/ that the positive eigennumbers c_i^2 correspond to the eigenvectors $\mathbf{e}_i$ of this matrix.

Assertion 2. The generalized solution $\mathbf{w}$ and $\mathbf{N}$ of problem (2.2) and (3.1) is determined by the formulas

$$\mathbf{w}(x_3, t) = \alpha_i c_i^{-1} q Y(t - x_3/c_i)\mathbf{e}_i, \quad q \equiv c_i/\alpha_i^2 \tag{3.3}$$
$$\mathbf{N}(x_3, t) = -\rho\alpha_i q Y'(t - x_3/c_i)\mathbf{e}_i$$

where α_i coefficients in the expansion of the vector $\mathbf{v}$ in the basis $\mathbf{e}_i$ and the prime denotes the derivative.

The proof is completely analogous to the proof of the corresponding assertion for an anisotropic linearly elastic medium /5, 9/.

Corollary. The force of body interaction with the medium is determined from the formula

$$P = -\rho q V(t) S(t) \tag{3.4}$$

where $S(t)$ is the area of the domain $G(t)$.

Indeed, the function $Y(t)$ in the problem of the vertical impression is the volume of the body under a cut at the height $H(t)$. Then taking account of (1.2) we obtain

$$Y'(t) = V(t)\, S(t) \tag{3.5}$$

We obtain (3.4) from (2.3), (3.3) and (3.5).

We note that dynamical contact problems for bodies with initial stresses were still examined only in the plane case /11/.

If the problem of the vertical impact of a body of mass m on a half-space surface with initial stresses is considered, the body velocity after the collision is determined exactly as in the problem of collision with an acoustic medium /12/. In particular, if the body is an elliptical paraboloid, i.e., $f(x_1, x_2) = Ax_1^2 + Bx_2^2$, $B > A > 0$, then the body velocity after the collision is determined from the formula

$$V(t) = 4V_0 \frac{e^{\beta t}}{(e^{\beta t} + 1)^2}, \quad \beta = \left(\frac{2\pi\rho q V_0}{m\sqrt{AB}}\right)^{1/2} \tag{3.6}$$

where V_0 is the body initial velocity. This formula (3.6) is exact as long as (1.3) is satisfied.

4. *Impression into a viscoelastic medium.* Let the medium occupying the half-space $\mathbf{R}_+^3$ be linearly viscoelastic. We will write the expressions for the integral characteristics of the solution in this case.

Here and henceforth we shall seek a generalized solution among piecewise-smooth functions /13/. It can be shown that the generalized piecewise-smooth solution of dynamic problems are among the weak solutions.

Maxwell medium. The governing relationships for this medium have the form

$$\sigma(t) = 3K\varepsilon(t) - \frac{3K}{t_v}\int_0^t \varepsilon(\lambda)\exp\left(\frac{\lambda - t}{t_v}\right)d\lambda \tag{4.1}$$

$$\sigma_{ij}^D(t) = 2\mu\varepsilon_{ij}^D(t) - \frac{2\mu}{t_s}\int_0^t \varepsilon_{ij}^D(\lambda)\exp\left(\frac{\lambda - t}{t_s}\right)d\lambda$$

$$\sigma \equiv \sigma_{kk}/3, \ \varepsilon \equiv \varepsilon_{kk}/3, \ \sigma_{ij}^D \equiv \sigma_{ij} - \delta_{ij}\sigma$$
$$\varepsilon_{ij}^D \equiv \varepsilon_{ij} - \delta_{ij}\varepsilon$$

where K and μ are the volume and shear elastic moduli, respectively, while t_v and t_s are the volume and shear relaxation times, respectively. Substituting these expressions into (2.1) and the equation of motion, we obtain equations for w_1 and w_2

$$w_i^{\cdot\cdot} + w_i^{\cdot}/t_s - (\mu/\rho)\, w_{i,33} = 0, \ i = 1, 2$$

whose solution taking boundary conditions into account and by virtue of uniqueness is $w_1(x_3, t) = w_2(x_3, t) = 0$.

In the case $t_s = t_v$ the integral displacement w_3 satisfies the equation

$$w_3^{\cdot\cdot} + w_3^{\cdot}/t_s - c_0^2 = 0,\ c_0^2 \equiv (4\mu/3 + K)/\rho$$

whose solution taking the boundary conditions into account has the form /14/

$$w_3(x_3, t) = h(t-\tau)\left\{Y(t-\tau)\exp\left(-\frac{\tau}{2t_s}\right) + \right.$$
$$\left. + \frac{\tau}{2t_s}\int_\tau^t \frac{Y(t-\lambda)}{\sqrt{\lambda^2-\tau^2}}\exp\left(-\frac{\lambda}{2t_s}\right) I_1\left(\frac{\sqrt{\lambda^2-\tau^2}}{2t_s}\right)d\lambda\right\},\quad \tau = \frac{x_3}{c_0}$$

where h and I_1 are the Heaviside and Bessel functions, respectively, while τ is the time during which the perturbation arrives at the point with coordinate x_3.

Hence, and from (4.1) we obtain the expression

$$\partial w_3(0, t)/\partial x_3 = c_0^{-1}h(t)\,[-Y'(t) - Y(t)/(2t_s) + Q(t)]$$
$$P(t) = -\rho c_0\,[Y'(t) + Y(t)/(2t_s) - Q(t)]$$
$$\Sigma_{33}(x_3, t) = \rho c_0^2\left[\frac{\partial w_3}{\partial x_3} - \frac{1}{t_s}\int_0^t \frac{\partial w}{\partial x_3}(x_3, \lambda)\exp\frac{\lambda - t}{t_s}\,d\lambda\right]$$
$$Q(t) = \frac{1}{2t_s}\int_0^t \frac{Y(t-\lambda)}{\lambda} I_1\left(\frac{\lambda}{2t_s}\right)\exp\left(-\frac{\lambda}{2t_s}\right)d\lambda$$

Voigt medium. In this case the relaxation between the stresses and strains can be written in the form

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \delta_{ij}\varepsilon_{kk}(K - 2\mu/3) + 2\mu t_s\varepsilon_{ij}^{\cdot} + (Kt_v - 2\mu t_s/3)\,\varepsilon_{kk}^{\cdot}\delta_{ij} \tag{4.2}$$

where $t_s^{\cdot}$ and t_v are the lag times under shear and bulk compression, respectively. The equation of plane wave propagation in this medium has the form

$$(K + 4\mu/3)\,w_{3,33} + \nu\dot{w}_{3,33} - \rho w_3^{\cdot\cdot} = 0,\ \nu = Kt_v + 4\mu t_s/3$$

and its solution taking the initial and boundary conditions (2.2) into account is written as follows /15/

$$w_3(x_3, t) = \int_0^t Y'(\lambda)\,\psi(x_3, t-\lambda)\,d\lambda$$
$$\psi(x_3, t) = \int_\xi^\infty I_0[2\sqrt{\xi(\lambda-\xi)}]\left\{\frac{1}{\sqrt{\pi\alpha}}\exp\left(-\frac{\lambda^2}{4\alpha} - \alpha\right) + \right.$$
$$\left.\frac{1}{2}\left[e^{-\lambda}\operatorname{erfc}\left(\frac{\lambda}{2\sqrt{\alpha}} - \sqrt{\alpha}\right) - e^{\lambda}\operatorname{erfc}\left(\frac{\lambda}{2\sqrt{\alpha}} + \sqrt{\alpha}\right)\right]\right\}d\lambda$$
$$\alpha \equiv \frac{\rho c_0^2 t}{\nu},\quad \xi = \frac{\rho c_0 x_3}{\nu}$$

The expression for the contact interaction force,taking (2.2) and (4.2) into account, has the form

$$P(t) = -\rho\,[c_0^2 w_{3,3}(0, t) + \nu\dot{w}_{3,3}(0, t)/\rho]$$

A homogeneous linear heriditary-elastic medium. We will assume that the orthotropy axes coincide with the coordinate axes.

The type of boundary conditions in the dynamical problem (2.2) does not change throughout the whole extent of the process; consequently, the Volterra correspondence principle /16/ holds for the problem under consideration. Starting from this principle, we take the expression for the force of body interaction with the orthotropic elastic medium /9/ and replace the function of the elastic constants (the wave velocity a is along the x_3 axis) by an appropriate operator $(\{a + A^*\}$ where A^* is the Volterra operator with kernel $A(t)$). Then we obtain for the contact interaction force

$$P(t) = -\rho\left[aY'(t) + Y(t)A(0) + \int_0^t Y(t-\lambda)A'(\lambda)\,d\lambda\right]$$

The governing relationships for the medium under consideration can be written in the form /16/

$$\sigma_{ij} = \left[E_{ijkl}\varepsilon_{kl}(\mathbf{x}, t) - \int_0^t \Gamma_{ijkl}(t-\lambda)\,\varepsilon_{kl}(\mathbf{x}, \lambda)\,d\lambda\right]$$

where E_{ijkl} is the instantaneous elastic moduli tensor and Γ_{ijkl} is the kernel of the Volterra operator. Then $a = \sqrt{E_{3333}/\rho}$ for an orthotropic medium.

The reasoning presented above repeats the reasoning used earlier when investigating wave propagation in isotropic viscoelastic rods /17/.

5. ***Isotropic elastic medium inhomogeneous through the depth.*** The governing relations for this medium have the form

$$\sigma_{ij} = K(x_3)\,\varepsilon_{kk}\delta_{ij} + 2\mu(x_3)\,\varepsilon_{ij}$$

If integral characteristics of solutions of the problem about impression in the medium under consideration are introduced by using (2.1), then we obtain that $w_1 \equiv w_2 \equiv 0$ and w_3 satisfies the equation

$$\frac{\partial}{\partial x_3}\left[E(x_3)\frac{\partial}{\partial x_3}w_3\right] - \rho(x_3)\,w_3^{\cdot\cdot} = 0, \quad E(x_3) \equiv \left[K(x_3) + \frac{4}{3}\mu(x_3)\right] \tag{5.1}$$

The perturbed domain in the initial-boundary value problem obtained for a plane wave is separated from rest by a wave front surface that goes from the point 0 to the point x_3 in the time $\tau(x_3)$. As we know, the function $\tau(x_3)$ satisfies the eikonal equation /18/ from which we find

$$\tau(x_3) = \int_0^{x_3}\sqrt{\frac{\rho(\lambda)}{E(\lambda)}}\,d\lambda \tag{5.2}$$

Following the ray method /18/, we seek the solution of (5.1) in the form

$$w_3 = \sum_{n=1}^{\infty}\Phi_n(x_3)\frac{[t-\tau(x_3)]^n}{n!} \tag{5.3}$$

The sum starts with $n = 1$ since we consider the function w_3 to be continuous, while its first derivatives may undergo discontinuity at the front.

We note that it follows from relationship (5.3) that

$$w_{3,3} = -\tau' w_3^{\cdot} + \sum_{n=1}^{\infty}\Phi_n'(x_3)\frac{[t-\tau(x_3)]^n}{n!} \tag{5.4}$$

Hence, taking into account (5.2) for the force of body interaction with the medium to obtain

$$P(t) = -E(0)\,w_{3,3}(0) = \rho(0)\sqrt{\frac{E(0)}{\rho(0)}}\,Y'(t) - H(t),$$

$$H(t) = E(0)\sum_{n=1}^{\infty}\Phi_n'(0)\frac{t^n}{n!}$$

Let k be the number of the first derivative of the function Y that does not vanish at $t = 0$. Then the term $\tau' w_3^{\cdot}(0)$ on the right-hand side of (5.4) is of order t^{k-1} while the term $H(t)$ is of the order t^k. We hence conclude that the first of these terms is of an order greater than the second for small t.

Thus, the expression for the interaction force is naturally identical with the expression for this same force in a homogeneous medium with density $\rho(0)$ and modulus $E(0)$ apart from terms of lower order of smallness in t.

We will show how to evaluate the correction terms $\Phi_n'(0)\,t^n/n!$. To do this we substitute the series (5.3) into (5.1) and equate coefficients for $(t-\tau)^n$ to zero. Taking into account that the function τ satisfies the eikonal equation, we obtain a system of equations in the functions Φ_n

$$\Phi_1' + \Phi_1(E'/E + \rho'/\rho)/4 = 0 \tag{5.5}$$

$$\Phi_{n+1}' + \Phi_{n+1}(E'/E + \rho'/\rho)/4 = \Phi_n''E/[2(\rho E)^{1/2}] + \Phi_n'E'$$

$$\Phi_n(0) = Y^{(n)}(0)$$

The last equality results from the condition $w_3(0, t) = Y(t)$ and the relationship (5.3). In the problem under consideration $Y(0) = Y'(0) = 0$. Let $Y^{(k)}(0) \neq 0$, $Y^{(k-1)}(0) = \ldots = Y'(0) = 0$. Then we obtain from system (5.5)

$$\Phi_1(x_3) \equiv \Phi_2(x_3) \equiv \ldots \equiv \Phi_{k-1}(x_3) \equiv 0$$
$$\Phi_k'(0) = -Y^{(k)}(0)(E'/E + \rho'/\rho)/4$$

Hence, it follows that the first correction term in the expression for the interaction force has the form

$$\Phi_k'(0)\, t^k/k! = -Y^{(k)}(0)\, t^k (E'/E + \rho'/\rho)/(4k!)$$

Differentiating the equation for Φ_k successively, we calculate the values of all the derivatives of the function Φ_k at the point 0. Substituting the derivatives found into the next expression, we find $\Phi_{k+1}'(0)$. Successively repeating the procedure described, we find all $\Phi_n'(0)$.

Remark. Different methods of investigating plane-wave propagation in inhomogeneous elastic and viscoelastic media are discussed in a number of publications (/17-20/, for example). It follows from the theorem proved above that these methods are applicable for investigating integral characteristics of the solutions of spatial problems about the dynamical impression of solid bodies into media, inhomogeneous in depth, with linear governing relationships.

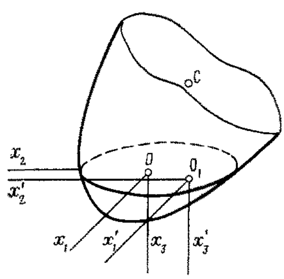

Fig.2

6. Impression with body rotation. The cases of vertical impression of a blunt body in different continuous media were examined above. Following /4/, we extend all the results presented above about the integral characteristics of solutions to the case when the body impression in the half-space occurs with rotation (Fig.2).

Let C be the centre of body mass and let the body angular velocity ω and the velocity vector of the centre of mass v^C be known.

As in the case of vertical impression, the integral of vertical displacements of points of the boundary plane $Y(t)$ at the superseismic stage of the impression process will equal the volume of the part of the body being inserted. For impression with rotation the velocities of points of the domain $G(t)$ will differ, and the change in volume $Y(t)$ during the time interval Δt will be determined from the formula

$$\Delta Y(t) = \iint\limits_{G(t)} v_3(x_1, x_2, 0, t)\, \Delta t\, dx_1 dx_2 \qquad (6.1)$$

We introduce the coordinate system $O_1x_1'x_2'x_3'$ (Fig.2) which is obtained by parallel transfer of the system $Ox_1x_2x_3$ in the $x_3 = 0$ plane to the point O_1, where O_1 is the projection of the point C on the boundary plane. Then we obtain according to Euler's theorem

$$v_3(x_1', x_2', 0, t) = v_3(0, 0, 0, t) + \omega_1 x_2' - \omega_2 x_1' \qquad (6.2)$$

and it follows from (6.1) and (6.2) that

$$Y'(t) = v_3^C S + \omega_1 S_1^* - \omega_2 S_2^*$$

where S_i^* is the static moment of the domain $G(t)$ about the x_i' axis.

Therefore, as for vertical impression and for impression with rotation, finding the integral characteristics of the solutions is equivalent to the problem of plane wave propagation in the linear medium under consideration.

The author is grateful to A.G. Khovanskii for discussing this paper.

REFERENCES

1. SKALAK R. and FEIT D., Impact on the surface of a compressible fluid, Trans. ASME, Ser.B., J. Eng. Industry, 88, 3, 1966.
2. SIMONOV I.V., The dynamic problem of the impression of an axisymmetric stamp into an elastic half-space. Inzh. Zh., Mekhan. Tverd. Tela, 2, 1967.
3. ROBINSON A.R. and THOMPSON J.C., Transient disturbances in a half-space during the first stage of frictionless identation of a smooth rigid die of arbitrary shape, Quart. Appl. Math., 33, 3, 1975.

4. GORSHKOV A.G. and TARLAKOVSKII D.V., Resultant reactions in the spatial problem of the impact of a solid on an elastic half-space. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 5, 1987.
5. BORODICH F.M., Dynamic contact of a blunt body with an anisotropic linearly elastic medium, Dokl. Akad. Nauk SSSR, 310, 1, 1990.
6. LADYZHENSKAYA O.A., Boundary-Value Problems of Mathematical Physics. Nauka, Moscow, 1973.
7. BROCKWAY G.S., On the uniqueness of singular solutions to boundary-initial value problems in linear elastodynamics, Archive Ration. Mech. and Analysis, 48, 3, 1972.
8. RUSHCHITSKII.YA.YA., On a method of studying the properties of a generalized solution in the theory of composite media, Prikl. Mekhan., 9, 2, 1973.
9. BORODICH F.M., Spatial problem of the impact of a blunt body on the surface of an elastic anisotropic half-space, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 4, 1990.
10. GUZ A.N., Elastic Waves in Bodies with Initial Stresses, 2, Propagation Regularities. Naukova Dumka, Kiev, 1986.
11. BABICH S.YU., On the investigation of a dynamic contact problem for a half-plane with Initial stresses, Prikl. Mekhan., 23, 4, 1987.
12. BORODICH F.M., On problem of the interaction of a blunt body with an acoustic medium, PMM, 52, 4, 1988.
13. PORUCHIKOV V.B., Methods of Dynamical Elasticity Theory, Nauka, Moscow, 1986.
14. DOETSCH G., Anleitung zum praktischen Gebrauch der Laplace-Transformation. R. Oldenbourg. Munchen, 1956.
15. ZVEREV I.N., Perturbation propagation in a viscoelastic and viscoplastic rod. PMM, 14, 3, 1950.
16. RABOTNOV YU.N., Elements of the Hereditary Mechanics of Solids. Nauka, Moscow, 1977.
17. FILIPPOV I.G., SHIRINKULOV T.SH., and MIRZAKABILOV S., Non-stationary Vibrations of Linear Elastic and Viscoelastic Media. Fan, Tashkent, 1979.
18. WHITHAM G.B., Linear and Non-linear Waves. J. Wiley, New York, London, 1974.
19. POBEDRYA B.E., Mechanics of Composite Materials, Izd. Moskov. Gosud. Univ. Moscow, 1984.
20. BREKHOVSKIKH L.M., Waves in Layered Media. Nauka, Moscow, 1973.

Translated by M.D.F.